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This Week's Citation Classic®

MacLane S. *Homology*. Berlin, FRG: Springer-Verlag, 1963.
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Homology is the algebraic process of describing connectivity and other qualitative aspects of topological spaces by means, first of Betti numbers, and then, by homology groups. Homological algebra, the subject of this book, uses the same techniques to develop homology and cohomology not for spaces, but for *algebraic* objects such as Lie algebras, groups, or associative algebras. Penetrating techniques such as spectral sequences and the bar construction are described. [The *SCI*® indicates that this book has been cited in more than 585 publications.]

Topology Applied to Groups and Algebras

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Topological questions were handled algebraically by H. Poincaré (1900), L.E.J. Brouwer (1910-1913), Heinz Hopf (1926), and others. About 1942, it developed that groups and other algebraic techniques could be used and, moreover, that topology in turn could be applied to algebra—giving "homological algebra." My own research in this field was almost all done in collaboration with Samuel Eilenberg, a refugee from Poland who taught at the University of Michigan, Indiana University, and extensively at Columbia University. Our first paper[1]—perhaps the first one in the then-new field—applied the known ideas of extensions of groups to finish a computation from Norman Steenrod about his homology of spaces; in a recent publication,[2] I have described the origins of this work in lectures that I was invited to give in 1941 at the University of Michigan. The first paper also led to our joint development of category theory.

Hopf (Zurich) wrote a decisive topological paper[3] indicating how the fundamental group of a space determined not only the first homology group, but also part of the second homology group of that space. Starting with his ideas, Eilenberg-MacLane developed the cohomology of groups,[4] the cohomology of spaces with only one homotopy group, and the (algebraic) bar construction used to calculate these. Our 15 joint papers—with other contributions by G. Hochschild, J.P. Hochschild-Serre, A. Heller, H. Cartan, and others—rapidly expanded this new field of homological algebra.

In 1956 Cartan and Eilenberg's[5] book codified the whole field (using Hopf's idea of resolutions) and showed how these techniques could be used to tackle purely algebraic problems in the theory of rings and of algebras. Their pioneering book seemed to me difficult in spots, so I resolved to write a more extensive exposition. This was based on courses that I had given at the University of Chicago, at Heidelberg, and at Frankfurt am Main. The finished book covered the whole array of ideas used: chain complexes in topology and algebra, the cohomology of groups with the Eilenberg-MacLane obstruction theory (later generalized by many authors), the construction of projective and injective resolutions, and the cohomology of algebras. The tensor products and their derived functors, the torsion products, were described, as was the bar construction, which has many uses. In topology, J. Leray, T.P. Serre, and Cartan had introduced the powerful computational technique of spectral sequences to which my student Roger Lyndon had also contributed; my book contained (in chapter XI) a clear and careful exposition of this technique—an exposition often used by young mathematicians learning the subject.

Homology has become a standard reference for this field, for its applications to group theory, ring theory, topology, and elsewhere. I believe that it was carefully and clearly written and that it appeared at about the right time, when the subject had reached a certain mature state, ready for applications. It is then not surprising that this exposition is a frequent choice for mathematicians needing to cite a source for these techniques. (There have been a number of other book expositions of the subject, including one by that polymath, Nicolas Bourbaki.)

Recently, the whole subject of homological algebra has displayed a new spirit, through the use of a certain "cyclic homology"[6]—which uses the bar construction and other techniques of homology with an ingenious new twist. I did not foresee this; I welcome it.

1. **Eilenberg S & MacLane S.** Group extensions and homology. *Ann. Math.* 43:757-831, 1942. (Cited 35 times since 1945.)
2. **MacLane S.** Group extensions after 45 years. *Math. Intell.* 10:29-35, 1988.
3. **Hopf H.** Fundamentalgruppe und zweite Bettische gruppe (Fundamental groups and second Betti groups). *Comment. Math. Helv.* 14:257-309, 1941. (Cited 20 times since 1945.)
4. **Eilenberg S & MacLane S.** Cohomology theory in abstract groups. I & II. *Ann. Math.* 48:51-78; 326-41, 1947. (Cited 80 and 70 times, respectively.)
5. **Cartan H & Eilenberg S.** *Homological algebra.* Princeton, NJ: Princeton University Press, 1956. (Cited 1,310 times.)
6. **Loday J L.** Cyclic homology, a survey. *Geometry and algebraic topology.* Warsaw, Poland: Banach Center, 1986. p. 281-3. Publication 18.

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