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Pian T H H. Derivation of element stiffness matrices by assumed stress distributions. *AIAA J.* 2:1333-6, 1964.
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A new method for deriving finite element stiffness matrices is based on the approximation of two independent fields: equilibrating stresses with undetermined parameters in the element, and boundary displacements in terms of modal displacements. Rectangular membrane elements are derived using different numbers of stress terms. [The *SCI*® indicates that this paper has been cited in over 85 publications since 1964.]

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"When the finite element method was first introduced in 1954 for aircraft structural analysis,[1,2] stiffness matrices for the membrane elements were derived from a potential energy principle that involves one field, the assumed displacements. When the method was extended to plate bending analysis that required C^1 continuity for the lateral displacement, considerable difficulties were encountered. By 1964, several investigations were made on alternative methods that involved multifield variational principles. The method which I proposed in this paper was to derive element stiffness matrices by complementary energy principles using assumed equilibrating stresses in the element and displacements along the element boundary. This is a method that can be most conveniently adopted to fit the existing finite element analysis programs, hence it is most attractive among the multifield finite element methods. The method is now known as the assumed stress hybrid method.[3-5]

"In nearly 20 years, hybrid elements have been used for both solid and fluid linear as well as geometric and material nonlinear problems. The method has been demonstrated to be advantageous even for problems that require only C^0 continuity conditions. The method can be used to construct special elements with stress singularities. Hybrid elements can also be constructed to avoid locking phenomena which may appear in the assumed displacement method under conditions of constraint such as incompressibility, etc. Hybrid elements are now used in many commercial finite element computer programs. At this time, there still are many researchers working on the improvement and applications of the assumed stress finite element method. For example, in certain cases the stresses need to satisfy the equilibrium conditions in each element only in a variational sense.

"I first thought of using the multifield approach for constructing finite element stiffness matrices when I was preparing a lecture one evening in January 1964 for a graduate subject on variational and matrix methods in structural mechanics. I used the Reissner principle, which contains element stresses and displacements as field variables. After the stiffness matrix of a rectangular membrane element had been obtained, I suddenly realized that, since my assumed stresses satisfied equilibrium conditions, I could have obtained the same result using the complementary energy principle and hence could avoid the construction of displacement functions in the element. The paper was published as a short technical note in *AIAA Journal* less than six months from the date when I prepared that lecture, and within four months after I completed the manuscript. Within a year, I showed how the method could be applied to C^1 place bending problems,[6] as did R.T. Severn and his colleagues in Bristol, England.[7]

"For this contribution, in 1974, I was awarded the von Karman Memorial Prize for outstanding contributions to aerospace structural material technology in the past decade, sponsored by TRE Corporation, Beverly Hills, California. In 1975, I received from the American Institute of Aeronautics and Astronautics, the Structures, Structural Dynamics, and Materials Award partly for this innovative development of the assumed stress finite element model."

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2. **Argyris J H.** *Energy theorems and structural analysis.* London: Butterworth, 1960. 85 p.
3. **Pian T H H.** Hybrid models. (Fenves S J, Robinson A R & Schnobrich W C, eds.) *Numerical and computer methods in structural mechanics.* New York: Academic Press, 1973. p. 50-78.
4. ----------------. Variational and finite element methods in structural mechanics. *Isr. J. Technol.* 16:23-33, 1978.
5. ----------------. Variational and finite element methods in structural mechanics. *RCA Rev.* 39:648-64, 1978.
6. ----------------. Element stiffness matrices for boundary compatibility and for prescribed boundary stresses. (Przemieniecki J S, Bader R M, Bozich W F, Johnson J R & Mykytow W J, eds.) *Matrix methods in structural mechanics.* Springfield, VA: National Technical Information Service, 1966. NTIS No. 646300.
7. **Severn R T & Taylor P R.** The finite element method for flexure of slabs when stress distributions are assumed. *Proc. Inst. Civil. Eng.* 34:153-70, 1966.

