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This Week's Citation Classic®

Arthurs A M. *Complementary variational principles.*
Oxford, England: Clarendon Press, (1970) 1980. 154 p.
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The monograph describes the variational theory of global dual extremum principles and its applications in mathematical physics. The underlying structure is canonical in which the Hamiltonians are saddle functions. A single generalized action functional on "phase space" generates related functionals in the component spaces. These, in turn, are the source of dual maximum and minimum principles providing under wide conditions upper and lower bounds for the action and error estimates for variational solutions. Many applications to linear and nonlinear boundary value problems are included. [The *SCI®* indicates that this book has been cited in more than 190 publications.]

Canonicals, Saddles, and Bounds

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Looking back, I now see that my introduction to variational principles was a most fortunate one. It began with Alex Dalgarno (now at Harvard) teaching in my undergraduate course about the elegant ideas of Hamilton's principle in dynamics. This was coupled with the pleasure of discovering Goldstein's classic book,[1] and continued with Benno L. Moiseiwitsch (nephew of the famous concert pianist) taking us into the realm of quantum scattering theory, with results described later in his impressive book.[2]

During this period, I suppose, the seeds of one of my major research interests were sown. I moved to Oxford in 1960 and, through Handel Davies, became interested in Feynman integrals, which lie at the heart of representing quantum mechanics directly in terms of the action.[3] These studies led to a year at MRC, Wisconsin, where the basic elements of a coherent theory of complementary variational principles had recently been worked out by Ben Noble.[4] There, in June 1966, I was lucky enough to hear Louis Rall give a brilliant exposition of the main ideas. That talk revealed to me whole new aspects of the elegance and power of Hamiltonian theory. Thus began my involvement with the subject of dual extremum principles. I immediately started to think about its implications and development in the context of mathematical physics. The early work was associated with linear boundary value problems. Quite soon, though, nonlinear problems were tackled, at first in terms of a rather restricted local theory (the first edition of this book, 1970) and later using global analysis (the second edition, 1980).

To turn a boundary value problem into a variational problem, we need to find a functional (the action) that is stationary at the solution of the boundary value problem. Dual extremum principles, when they hold, then lead in a systematic way to upper and lower bounds for the action. In many applications, the action is of interest since it can provide a measure of physical properties as various as absorption probabilities, torsional rigidity, electromagnetic energy, and scattering lengths. In other cases the focus shifts to the solution of the boundary value problem.

Early ideas of dual principles go back to the turn of the century, to Rayleigh in acoustics and Thomson in electrostatics. The first systematic approach, forerunner of the present method, was introduced by Friedrichs in 1929 and employed canonical and involutory transformations. Another seemingly unrelated approach was introduced by J.L. Synge in 1946 as the method of the hypercircle,[5] which applies to certain linear problems. The hypercircle caused me some difficulties at first. I had read Courant and Hilbert on the subject and contacted Synge about their version in the early 1970s. Synge replied that he did not recognize his own baby, and then generously proceeded to put me right. A busy correspondence ensued that led me to establish the equivalence of the hypercircle and the canonical variational method.[6]

Perhaps the reason why my book has been cited comes down simply to the fact that, for a long time, it was the only book on the subject. Now that has all changed with the arrival of Michael J. Sewell's monograph,[7] that paints a broader canvas and which I feel sure will be extremely successful.

1. **Goldstein H.** *Classical mechanics.* Cambridge, MA: Addison-Wesley, 1950. 399 p. (Cited 1,890 times.) [See also: **Goldstein H.** Citation Classic. *Current Contents/Engineering, Technology & Applied Sciences* 12(2):16, 12 January 1981. Reprinted in: *Contemporary classics in engineering and applied science.* (Thackray A, comp.) Philadelphia: ISI Press, 1986. p. 47.]
2. **Moiseiwitsch B L.** *Variational principles.* New York: Interscience, 1966. 310 p. (Cited 80 times.)
3. **Davies H.** Hamiltonian approach to the method of summation over Feynman histories. *Proc. Camb. Philol. Soc.* 59:147-55, 1963.
4. **Noble B.** Complementary variational principles for boundary-value problems. I. Basic principles with an application to ordinary differential equations. University of Wisconsin Mathematics Research Center Report No. 473, 1964.
5. **Synge J L.** *The hypercircle in mathematical physics.* Cambridge, England: University Press, 1957. 424 p. (Cited 150 times.)
6. **Arthurs A M.** On variational principles and the hypercircle for boundary value problems. *Proc. Roy. Irish Acad. Sect. A* 77:75-83, 1977.
7. **Sewell M J.** *Maximum and minimum principles.* Cambridge, England: University Press, 1987. 468 p.

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