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Gilkey P B. The spectral geometry of a Riemannian manifold.
J. Differen. Geom. 10:601-18, 1975.
[University of California, Berkeley, CA]

Let M be a compact Riemannian manifold without boundary and let D be a self-adjoint elliptic second order operator with scalar leading symbol. As $t \to 0$, $Tr(\exp(-tD)) \approx \Sigma_n a_n(D) t^{(n-m)/2}$. The heat equation asymptotics $a_n(D)$ are computed for $n \leq 6$ in terms of geometrical data. [The *SCI*® indicates that this paper has been cited in over 155 publications, making it the most-cited paper for this journal.]

Hearing the Shape of a Drum

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One wants to know the extent to which the spectrum of the Laplacian controls the geometry of the manifold. M. Kac originally put the question; it has been restated subsequently by M. Protter: "Suppose a drum is being played in one room and a person with perfect pitch hears but cannot see the drum. Is it possible for her to deduce the precise shape of the drum just from hearing the fundamental tone and all the overtones?"[1]

Let D be a second order operator with scalar leading symbol; as $t \to 0$, $Tr(\exp(-tD)) = \Sigma_n a_n(D) t^{(n-m)/2}$. The heat equation asymptotics $a_n(D)$ are locally computable invariants of the spectrum of M. My paper was an attempt to compute these asymptotics in a very general context that would include the Laplacian on forms, on spinors, etc. The invariants a_n vanish if n is odd for $\partial M = \emptyset$; the invariants a_0, a_2, and a_4 are fairly easy to compute by hand. I used extensive computer calculations to compute a_6. Recently, a_8 has been computed by I.G. Avramidi[2] and independently by P. Amsterdamski, A. Berkin, and D. O'Connor;[3] it has formidable combinatorial complexity. This gives complete information concerning a_n for $n \leq 8$; there is partial information available for all n (see reference 4 for details). If $\partial M \neq \emptyset$, the situation is somewhat more complicated; recent work with Tom Branson[5] computes a_n for $n \leq 4$.

The formulas become exponentially more complicated as n increases; for example, the formula for a_6 has 46 terms if $\partial M = \emptyset$, while the formula for a_4 if $\partial M = \emptyset$ has over 50 terms. There are by now many different algorithms for computing the heat equation asymptotics; there always seems to be an irreducible combinatorial complexity.

After doing the calculation of a_4, I found a paper by T. Sakai[6] that did the scalar case using different methods. This enabled me to check directly 17 of the coefficients. I recall the feeling of anxiety as I spent one entire afternoon comparing the two answers; this was a nontrivial calculation as we had used different bases for the space of invariants. I made the final calculation to determine the two answers agreed! I looked out of my office in Princeton and just gazed at the view for a long time in great relief.

The paper on spectral geometry contains a sign error on page 609; Theorem 2.1 should read $D \approx D_\nabla - E$ (not $D_\nabla + E$). The error doesn't propagate and is isolated. Judging by the number of citations, the paper has been useful to lots of people. That is a very satisfactory payment for the very lengthy complicated calculations involved. I am currently embarked on studying first order operators and operators with nonscalar leading symbol, which I hope will prove equally useful!

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1. **Protter M.** Can one hear the shape of a drum? Revisited. *SIAM Rev.* 29:185-97, 1987.
2. **Avramidi I G.** The covariant technique for the calculation of the heat kernel asymptotic expansion. *Phys. Lett. B* 238:92-7, 1990.
3. **Amsterdamski P, Berkin A & O'Connor D.** b_8 'Hamidew' coefficient for a scalar field. *Class. Quantum Gravity* 6:1981-91, 1989.
4. **Gilkey P.** Leading terms in the asymptotics of the heat equation. *Contemp. Math.* 73:79-85, 1988.
5. **Branson T & Gilkey P.** The asymptotics of the Laplacian on a manifold with boundary. *Commun. Part. Diff. Equat.* 15:245-72, 1990.
6. **Sakai T.** On eigenvalues of Laplacian and curvature of Riemannian manifold. *Tohoku Math. J.* 23:589-603, 1971. (Cited 10 times.)