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The hard-hexagon model in lattice statistics is solved exactly. It undergoes a continuous fluid-to-solid phase transition, with critical exponents $\alpha=1/3$, $\beta=1/9$. The mathematics naturally involves the Rogers-Ramanujan and related identities. [The *SCI*® indicates that this paper has been cited in over 170 publications.]

Numerics, Conjectures, and Exact Results

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The hard-hexagon model is a good model of a two-dimensional fluid-solid phase transition, e.g., helium adsorbed onto graphite.[1] It turns out that the model can be solved exactly, for all densities. Its critical behaviour can be examined and compares well with experiment.

The basic mathematical problem is to calculate the grand partition function Ξ as a function of the activity z. The way this came to be done is a fascinating example of how numerical work, conjectures, and analysis can all complement one another. In 1967 David Gaunt of King's College, London, noticed that $\Xi(z)$ appeared to have two singularities in the complex z-plane, at points z_1 and z_2, where $z_1 + z_2 = 11$ and $z_1 z_2 = -1$. These were approximate numerical results, but perhaps these integer values were correct. If so, then the physical singularity would be at $z_1 = [(1 + \sqrt{5})/2]^5 = 11.090...$, i.e., the fifth power of the golden number.

Unfortunately, in those prudish days, one did not do this sort of thing in public, so this observation gathered dust in his notes. The next development was not for another 11 years, when B.D. Metcalf and C.P. Yang looked at the special case $z=1$. To four figures they found $\log \Xi = 0.3333$. Less inhibited, they speculated publicly that perhaps it was exactly $1/3$. Intrigued, my student Shiu Kuen Tsang (now Tin) and I made our own calculations, using corner transfer matrices[2] and their eigenvalues. This is a powerful method: we soon found $\log \Xi = 0.333242721976....$

This killed conjecture number two; but I noticed a curious fact: ratios of eigenvalues were integer powers of some common number x, as in the previously solved eight-vertex model (8VM). Did this mean the hard hexagon model was also solvable? First I did some series expansions for arbitrary z. Guided by the 8VM, I expanded (in the fluid phase) z as a function of x, in the product form $z = -x(1-x)^{c_1}(1-x^2)^{c_2}(1-x^3)^{c_3}....$ I hoped for a simple repeat pattern and indeed found $c_1,...,c_{29}$ were 5,-5, -5,5,0, 5,-5,-5,5,0, 5,-5,-5,5,0, 5,-5,-5,5,0, 5,-5,-5,5,0, 5,-5,-5,5! Encouraged, I tried the algebraic tricks that had worked for the 8VM, first looking for a "star-triangle" relation. It all worked. There was a transition, exactly where Gaunt had conjectured.

More followed: cognoscenti may recognize the function $z(x)$ as one studied by Ramanujan. The algebra gave the density $\rho(x)$ as the ratio of two series, while the computer expansions suggested it was a simple product of the above type. I proved the equivalence of the two forms and found I had stumbled onto the famous Rogers-Ramanujan identities. Other properties and other phases suggested other, more complicated, identities. At this point I sent out an SOS, which was answered by several mathematicians, notably George E. Andrews. This led to a very fruitful collaboration and to the solution of the ABF models.[3] These immediately excited interest, being a realization of a large class of two-dimensional critical behaviour predicted by conformal invariance.[4]

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4. **Cardy J L.** Conformal invariance. (Domb C & Lebowitz J L, eds.) *Phase transitions and critical phenomena.* London: Academic Press, 1987. Vol. XI. p. 55-126. (Cited 40 times.)

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