Efron B. Bootstrap methods: another look at the jackknife. *Ann. Statist.* 7:1-26, 1979.
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The bootstrap, introduced here, uses large amounts of computation in place of traditional mathematical models (like the bell-shaped curve) to construct confidence intervals, standard errors, and other measures of statistical variability. Computationally intensive statistical inference is particularly useful in complicated inference problems, like map drawing, image reconstruction, or multivariate analysis, when mathematical modeling of error distributions is difficult. [The *SCI*® and *SSCI*® indicate that this paper has been cited in over 410 publications, making it the most-cited paper from this journal.]

Computer-Intensive Statistical Inference

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The following question occurred to statisticians in the 1950s, 1960s, and 1970s: Why, in the era of electronic computers, was everyone still using classical theories, like linear regression and analysis of variance, constructed for mechanical calculators? The Quenouille-Tukey jackknife, circa 1965, took an intriguing step beyond classical "bell-shaped curve" theory. It substituted a substantial amount of computation, perhaps 10 times that of the traditional methods, for most of the mathematical approximations usually involved in calculating biases and standard errors.

The jackknife attracted some theoretical attention, but disappointingly little practical use. It didn't even make it into elementary textbooks on nonparametric statistics, despite being nonparametric, elementary (to describe), and eminently useful.

My paper was intended to strengthen the jackknife by putting it in a firmer theoretical setting. The bootstrap, introduced for just that purpose, is essentially the oldest idea in the statistical book: substitute the empirical distribution of the data for the (unknown) true distribution in anything you wish to estimate. If what you wish to estimate is a bias or a standard error, then the jackknife suddenly appears as a straightforward linear approximation to the bootstrap.

In writing this down, and supporting the idea with numerical calculations, I saw that the bootstrap did well in its own right, usually outperforming the jackknife, applying to a wider variety of data structures, and always being easier to motivate and describe. The computational burden for the bootstrap, another factor or two of 10 beyond the jackknife, is tolerable in an era of fast and cheap computation. The straightforward nature of the bootstrap has been popular with statistical consumers. It is beginning to creep into the small circle of commonly used statistical techniques, with applications ranging from particle physics to econometrics. (The jackknife too is now more frequently used.) P. Diaconis and I[1] and R. Tibshirani and I[2] present a variety of applications.

Near the end of the original paper, Remark D raised a theoretical difficulty that bothered me greatly during the next decade. The question is one of statistical inference: How can one use bootstrap computations to construct accurate confidence intervals for an unknown parameter? Previous attempts to refine the jackknife in this direction had never shown substantial improvements over the classical confidence interval estimate $\pm$constant*(standard error).

A series of important papers by P. Bickel and D. Freedman,[3] K. Singh,[4] and others showed that the bootstrap is theoretically capable of an order of magnitude accuracy improvement over the classical intervals. In technical language, the bootstrap is capable of second order accuracy, compared to only first order accuracy for the classical intervals. My most serious attempt to make the bootstrap second order accurate in practice, as well as in theory, is "Better bootstrap confidence intervals."[5] The published discussion of this paper reviews several promising ways of constructing accurate bootstrap confidence intervals.

The 1979 paper ends with a joke concerning the name "bootstrap." The joke may have been on me since "bootstrap" doesn't translate into some languages. A recent communiqué from Professor M.T. Chao suggests the name "ladder in cloud jump" for the Chinese bootstrap.

1. **Diaconis P & Efron B.** Computer intensive methods in statistics. *Sci. Amer.* 248:116-30, 1983. (Cited 130 times.)
2. **Efron B & Tibshirani R.** Bootstrap methods for standard errors, confidence intervals, and other measures of statistical accuracy. *Statist. Sci.* 1:54-73, 1986. (Cited 60 times.)
3. **Bickel P & Freedman D.** Some asymptotic theory for the bootstrap. *Ann. Statist.* 9:1196-217, 1981. (Cited 55 times.)
4. **Singh K.** On the asymptotic theory of Efron's bootstrap. *Ann. Statist.* 9:1187-95, 1981. (Cited 70 times.)
5. **Efron B.** Better bootstrap confidence intervals. *J. Amer. Statist. Assn.* 82:171-200, 1987.

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