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This Week's Citation Classic®

Hashin Z. The elastic moduli of heterogeneous materials. *J. Appl. Mech.* 29:143-50, 1962; and **Hashin Z & Rosen B W.** The elastic moduli of fiber-reinforced materials. *J. Appl. Mech.* 31:223-32, 1964.
[Towne Sch. of Civil and Mechanical Engineering, Univ. of Pennsylvania, and Space Sciences Laboratory, General Electric Co., Philadelphia, PA]

In the first paper, bounds and expressions for the elastic moduli of two- or many-phase nonhomogeneous materials are obtained by an approximate method based on the variational theorems of the theory of elasticity and on a concentric-spheres model. In the second paper, bounds and expressions for the effective elastic moduli of materials reinforced by parallel, hollow, circular fibers are derived by a variational method. [The *SCI®* indicates that these papers have been cited in over 205 and 180 publications, respectively.]

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During 1959-1960 I spent a year at Harvard University as a research fellow, and I decided to tackle again, after many previous unsuccessful efforts, the difficult problem of evaluation of the effective elastic properties of a solid that contains spherical particles of another material. This problem and the related problems of the evaluation of effective conductivity and effective viscosity of a suspension of rigid spheres in a fluid had been of considerable interest for a long time; and in 1960 the only available exact results were for the case of dilute concentration (a few percent volume fraction of particles), in which case mutual interaction of particles can be neglected. Such results had been originated by very illustrious scientists: Maxwell for conductivity in 1873 and Einstein for viscosity in 1906.

I constructed a special geometrical model that I called composite spheres assemblage (CSA). Each composite sphere consists of a spherical particle that is surrounded by a concentric matrix shell. The volume fractions of particle and matrix material are the same in each sphere, but the spheres themselves can be of any size. An arbitrary volume is then filled out with composite spheres of diminishing size, and the CSA is approached as a limit of complete filling. This model was analyzed by variational methods, and it turned out that it yielded a closed-form exact solution for the effective bulk modulus and close lower and upper bounds for the effective shear modulus.

As is often the case, the development of my thinking was not too orderly. I started work with a single composite sphere and noted that it had the same effective bulk modulus for displacement and stress boundary conditions but a different effective shear modulus for pure shear displacement and pure shear tractions on the boundary. I found the latter results very confusing until I realized that on the basis of variational arguments they could be interpreted as upper and lower bounds for a composite material that consists entirely of composite spheres.

About a year later, at the University of Pennsylvania, I was involved, together with S. Shtrikman, in the construction of bounds for effective elastic properties of macroscopically isotropic, two-phase materials when the internal phase geometry is arbitrary and only the volume fractions are specified. This work has been published[1] and was the subject of a previous *Citation Classic* commentary. It turned out that the bulk modulus bounds could be identified with CSA exact bulk moduli, which led to the important conclusion that the arbitrary phase geometry bounds were the best possible when only phase volume fraction information was given. Bound improvement required more geometrical information. As for the shear modulus, it turned out that one of the arbitrary phase geometry bounds was always better than one of the CSA bounds, and this led to substantial improvement of the CSA bounds.

In 1962 I became involved in research on properties of unidirectional fiber composites, which was then an emerging technology, as a consultant to the Valley Forge Space Sciences Laboratory of General Electric. Since fibers are long circular cylinders, it was a natural step to treat this problem in terms of a composite cylinder assemblage, and this was done together with B.W. Rosen. However, such a material is anisotropic and has five independent effective elastic properties as opposed to the two of a spherical particle composite. We were able to obtain closed-form solutions for four of these while the fifth, the transverse shear modulus, could again only be bracketed by bounds that were, however, quite close.

Composite materials are of considerable practical importance, and there have been many attempts to evaluate effective properties on the basis of various assumptions and empiricisms. I think that the papers under discussion have been widely quoted because they give exact and simple closed-form results, though for special models of composite materials, that are mostly in very good agreement with experimental data.

For a recent review of work in this field, see reference 2.

1. **Hashin Z & Shtrikman S.** A variational approach to the theory of the elastic behaviour of multiphase materials. *J. Mech. Phys. Solids* 11:127-40, 1963. (Cited 350 times.) [See also: **Hashin Z.** Citation Classic. (Thackray A, comp.) *Contemporary classics in physical, chemical, and earth sciences.* Philadelphia: ISI Press, 1986. p. 344.]
2. **Bert C W & Kline R A.** Composite-material mechanics: properties of planar-random fiber composites. *Polym. Composite.* 6:133-41, 1985.

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