This Week's Citation Classic®

Hénon M & Heiles C. The applicability of the third integral of motion: some numerical experiments. *Astronomical J.* 69:73-9, 1964.
[Princeton University Observatory, NJ]

Motivated by the problem of the existence of a third integral of galactic motion, we investigated two simple dynamical systems: a two-dimensional potential and a mapping. Phase space was found to be divided into two regions with sharply different properties: in one region the orbits are very regular and intersect a surface of section along a smooth curve, while in the other region the orbits are very irregular ("chaotic" in present-day language) and the points of intersection are randomly scattered. [The *SCI*® indicates that this paper has been cited in over 500 publications, making it the most-cited paper for this journal.]

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In 1962, having completed a PhD on the dynamics of globular clusters, I was invited by Lyman Spitzer to the Princeton University Observatory for one year. An intriguing problem at that time was the hypothetical "third integral" for the motion of a star in an axisymmetric galaxy. The prevalent belief was that this integral did in fact exist, on the basis of several pieces of evidence: observations of nearby stars, numerical computations by A. Ollongren,[1] and a theory by G. Contopoulos.[2] However, nobody had been able to exhibit the integral in closed analytical form.

I thought that perhaps one would have a better chance of success by getting rid of the purely astronomical peculiarities of the problem (for instance, the model of the Galaxy then in use consisted of 13 superimposed ellipsoids!) and attacking it at a simpler, more fundamental level. So I started some computations with a simple fourth-order potential. I obtained well-behaved orbits, which once more seemed to confirm the existence of an additional integral. However, I was again unable to find the expression of that integral. Some orbits exhibited a slight fuzziness that puzzled me; it seemed to be a bit larger than what one could expect from numerical errors.

As a "visiting lecturer," I had to supervise a six-month research project by one of the Princeton graduate students. So I asked Carl Heiles to investigate another simple potential, of third order. Heiles wrote the program, ran it, and came back with astounding results. In some cases the star orbits were quite regular, in the usual way, but in other cases they behaved wildly, jumping here and there in an apparently random fashion. These results were hard to believe; the people who saw them, including us, were skeptical and wondered about a possible bug in the program. So we redid the computations independently, using another programmer (me), another program, another integration algorithm, and another computer. The same results emerged! So here we had a clear case where the "third integral" in fact did not exist.

By a fortunate coincidence V. Arnold and J. Moser, working independently, had at the same time obtained their proofs of what was to become famous as the *KAM theorem*. In December 1962 I attended a gathering of astronomers at Yale. Moser was present and gave an illuminating presentation of the latest mathematical results and their consequences for the dynamics of nonintegrable systems. Suddenly everything fell into place: qualitatively at least, the mathematical theory completely explained the strange mixture of order and chaos found in our numerical results.

So we went on to produce a paper. There were no plotting devices available at that time, and with the help of my young wife we spent some evenings plotting hundreds of points by hand on large sheets of graph paper. The initial title of the paper was to be "The hypothetical third integral...." However, to express such doubts was then heresy for some colleagues, and we had to change the title to a more diplomatic "The applicability of the third integral...."

In subsequent years, the number of papers reporting similar behavior in all corners of science increased, slowly at first and then more rapidly. The description was much refined. The phenomenon was variously characterized as "semi-ergodic," "irregular," "wild," "erratic," "stochastic," "aperiodic," "turbulent," "strange,"...until finally the word "chaotic" prevailed. This phenomenon has recently been reviewed by G. Iooss[3] and P.C.H. Martens.[4]

Why, in my opinion, did our paper receive a fair number of citations? The appearance of chaos in Hamiltonian systems had been previously observed, but only in specialized contexts, such as studies of particle accelerators.[5] Our paper may have been the first to call attention to the generality of the phenomenon by moving away from specialized applications and studying instead appropriately designed "model problems."

1. **Ollongren A.** Three-dimensional galactic stellar orbits. *Bull. Astron. Inst. Neth.* 16:241-96, 1962. (Cited 55 times.)
2. **Contopoulos G.** A third integral of motion in a galaxy. *Z. Astrophysik* 49:273-91, 1960. (Cited 90 times.)
3. **Iooss G, Helleman H G & Stora R,** eds. *Chaotic behaviour of deterministic systems.* Amsterdam: North-Holland, 1983. 708 p.
4. **Martens P C H.** Applications of non-linear methods in astronomy. *Phys. Rep.—Rev. Sect. Phys. Lett.* 115:315-78, 1984.
5. **Symon K R & Sessler A M.** Methods of radio frequency acceleration in fixed field accelerators with applications to high current and intersecting beam accelerators. (Regenstreif E, ed.) *Comptes-Rendus du Symposium du CERN sur les Accelerateurs de Haute Energie et la Physiques des Mesons.* 11-23 June 1956, Geneva, Switzerland. Geneva: CERN, 1956. Vol. 1. p. 44-58.

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