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Sandberg I W. Some results on the theory of physical systems governed by nonlinear functional equations. *Bell Syst. Tech. J.* **44**:871-98, 1965.
[Mathematics and Mechanics Research Center, Bell Laboratories, Murray Hill, NJ]

The paper presents results concerning the solutions of nonlinear functional equations that frequently arise in the study of physical systems. Considered in detail are conditions under which the solutions of a certain nonlinear integral equation have various properties of interest, such as boundedness. [The *SCI*® indicates that this paper has been cited in over 80 publications since 1965.]

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"In 1961, while a member of the Mathematics and Mechanics Research Center at Bell Laboratories, I became interested in using functional-analytic techniques to address general problems concerning nonlinear circuits and systems. Functional analysis, with its operator theory focus, seemed to be interesting and powerful, and to provide a natural framework within which significant system-theoretic questions might be formulated. At first, problems other than stability were considered. These concerned, for example, methods for recovering distorted signals, the evaluation of distortion in nonlinear feedback systems, the validity of the describing-function method, the mathematical basis for truncation techniques for the solution of the steady-state equations of nonlinear networks, and criteria for the causality of nonlinear operators.

"The subject of the paper discussed here is the introduction and development of input-output stability theory. This differs from Lyapunov stability theory, and is concerned primarily with the problem of determining criteria for the boundedness and continuity of the relations that express a dynamic nonlinear system's outputs in terms of its inputs. Stability theory is of wide general interest, partially because often the most important design constraint imposed on a system is that it be stable. The paper is cited frequently because it provides analysts with a very different approach and body of results than is associated with classical Lyapunov theory.

"It was the earlier work mentioned here (which involves normed linear spaces, Hilbert spaces, projection operators, and causality), on the one hand, and an initially unrelated stability study concerning differential equations, on the other, that led to the introduction of 'extended spaces.' These spaces were needed to address input-output stability questions, and they play a central role in the paper.

"The 'small gain theorem,' the 'passivity theorem,' and the 'circle criterion' (which is a graphical criterion involving a critical disk in the complex plane) are the three most well-known results concerning input-output stability theory. Several closely related versions of each are now available in the literature. With regard to the circle condition, the paper focuses attention on the key role the condition plays in connection with *several* types of input-output properties. For example, the paper's Theorem 4 provides conditions under which the response of a feedback system is ultimately T-periodic (by which is meant, basically, that it approaches a bounded periodic function of period T as $t \to \infty$) whenever its input is ultimately T-periodic. This relatively wide range of input-output properties associated with the circle condition illustrates the intent and potential of input-output approaches. Further details, including references to the related work of others, can be found in a recent paper.[1]

"Methods similar to those used to obtain the circle criterion were later employed to prove very early theorems concerning the properties of numerical integration techniques for nonlinear ordinary differential equations under finite stepsize hypotheses. (For references, see reference 1. The then-typical kind of result in the numerical analysis literature concerned only asymptotic properties associated with letting the stepsize approach zero.) Finally, we note that circle-criterion type results concerning the stability of the solutions of nonlinear *partial* differential equations have also been obtained. A particularly nice contribution is the paper by Banks[2] that contains additional references."

1. **Sandberg I W.** A perspective on system theory. *IEEE Trans. Circuit. Syst.* **31**:88-103, 1984.
2. **Banks S P.** The circle theorem for non-linear parabolic systems. *Int. J. Control* **34**:843-51, 1981.

