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This Week's Citation Classic

Twomey S. On the numerical solution of Fredholm integral equations of the first kind by the inversion of the linear system produced by quadrature. *J. Assn. Comput. Mach.* **10**:97-101, 1963. [US Weather Bureau, Washington, DC]

Quadrature formulas reduce integral equations of the first kind to systems of linear equations, which, although approximate, can be given quite high accuracy. Direct inversion of such linear systems, as a method of solving the Fredholm integral equations, was shown to be unstable and so to fail badly; incorporating a smoothness constraint into the solution process can restore stability and give acceptable solutions. [The *SCI®* indicates that this paper has been cited in over 160 publications since 1963.]

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"The cited paper was so far out of my mainstream of activity at the time it was written that it was the only manuscript that I—a mediocre typist at best—have ever typed myself. Needless to say, it was not very long or it would probably still be waiting to get finished.

"Some motivation for the work behind the paper may have been a subconscious guilt complex. Earlier, in my first few months of employment after leaving college, I had wanted to copy a fairly straightforward electrical test network; all the component values were not given, but it seemed very straightforward to solve a linear system of network equations for six unknown component values. It took me more than a day by desk calculator to obtain a completely useless answer—useless because some of the values were physically unachievable (e.g., negative resistances). I put the project aside before my boss woke up to the wastage of so much time, but it rankled a little that the answer so closely fitted the original set of equations, while it was physically nonsensical. I was sure that greater accuracy would have cured everything, so I missed the important point completely—that the inversion of linear systems, even if mathematically unique, can be exceedingly unstable. I did learn, and took to heart, not to mess with large (even six-by-six!) linear systems of equations with desk calculators.

"Comforted by the vaunted accuracy of computers, I was not at all apprehensive later on about inverting a 20 × 20 system of equations on the top computer of the time. My earlier experience—all but forgotten—came back in double-quick time when the 20 × 20 project crashed down around my ears in an even more spectacular fashion—nonsensical answers satisfying the original equations to one part in 10^8 or so, yet horribly different from the known correct answer for test cases on hypothetical data. With the computer, experimentation was almost painless, and this time understanding gradually came. The perils of instability and the practical equivalence of near-singularity and indeterminacy were becoming apparent when my 1963 *Journal of the Association for Computing Machinery* paper was written.

"The frequent citation of this paper can probably be attributed to timing. The theory of linear systems, least-squares solutions, eigenanalysis, quadratic surfaces—all are classical mathematics and go back to Gauss, Jacobi, and other founding fathers; but numerical application of many of their techniques had to await the modern computer and, in the interim, became regarded as 'conceptual' rather than enforceable, practical procedures. The literature in the early-1960s contained many suggestions for improving the accuracy of inversion algorithms. However, the fact was neglected that, even given perfect accuracy, inversions of near-singular systems had often quite extraordinary error-magnifying powers. Once that was recognized and its reason (which has nothing to do with imperfect accuracy) understood, treatment for what has been picturesquely called a 'pathological' condition was not too hard to find. What was needed, it turned out, was a rephrasal of the question asked because it was the question and not the answer that was wrong! A more detailed discussion can be found in *Introduction to the Mathematics of Inversion in Remote Sensing and Indirect Measurement*."[1]

1. Twomey S. *Introduction to the mathematics of inversion in remote sensing and indirect measurement.* New York: Elsevier, 1977. 243 p.