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This Week's Citation Classic

Hashin Z & Shtrikman S. A variational approach to the theory of the elastic behaviour of multiphase materials. *J. Mech. Phys. Solids* **11**:127-40, 1963.
[Univ. Pennsylvania, Philadelphia, PA and Weizmann Inst. Science, Rehovoth, Israel]

The effective elastic moduli of multiphase (composite) materials of arbitrary interface geometry are bounded from above and below in terms of simplest information: phase properties and phase volume fractions. Some of the bounds are best possible in terms of information used. Closeness of bounds depends on relative phase stiffness. [The *SCI*® indicates that this paper has been cited over 165 times since 1963.]

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"In 1961, when I was a faculty member at the College of Engineering of the University of Pennsylvania in Philadelphia, I had the good fortune to meet my former classmate from Technion –Israel Institute of Technology, S. Shtrikman, who was spending two years at the Franklin Institute Laboratories. It turned out that both of us were involved in the same research subject: prediction of effective properties of multiphase (composite) media in terms of phase properties and phase geometry. He was interested in electrical and magnetic properties while I was concerned with elastic properties (no doubt because he graduated in electrical engineering and I in civil engineering).

"At that time the literature on the subject was already considerable, but only a small number of rigorous results were available. These consisted of solutions for properties of composites described as matrix containing dilute concentration of spherical particles, effective bulk modulus for a special geometry –composite spheres assemblage–for all volume concentrations of spherical particles, and bounds for effective properties based on the classical variational principles. Many workers in the field did not recognize the fact that phase volume fractions are totally insufficient information for determination of effective properties, although W.F. Brown, Jr. of the University of Minnesota had shown in 1954 that effective electric properties of a two-phase medium depend on the entire statistics of the phase geometry.[1]

"Since, however, the statistics of phase geometry are never known in detail, a logical approach is to bound effective properties in terms of available geometrical information. This we did in terms of the simplest such information: phase volume fractions. The bounding procedure was based on new variational principles in elasticity, in terms of the stress polarization, which we had established. Previously we had treated in similar fashion the problem of magnetic permeability of multiphase materials.[2] The new variational principles for magnetostatics used for that problem had been developed by us in consequence of some ideas communicated to us by Brown.

"For elastic two-phase materials we were able to show that bounds for effective bulk modulus are best possible in terms of the geometrical information used (volume fractions). Thus bound improvement requires more geometrical information. We have not been able to show, even to this day, that the effective shear modulus bounds are also best possible though we suspect that this is so. The bounds are quite close when the ratio of elastic moduli of one phase to the other is not extreme (up to about ten).

"When we finished this work in 1961 we submitted it for publication to a prestigious applied mechanics journal in the US. It was ignominiously rejected. The reviewer, described to us as an 'outstanding authority,' had classified the paper as 'ramblings.' We subsequently sent it to the *journal of the Mechanics and Physics of Solids* which accepted it at once. This may appear funny today, but we failed to appreciate the humor of the situation in 1962.

"I believe the reason for frequent citation of the paper is that it established permanent results for general two-phase materials, namely, best bounds in terms of easily available information for arbitrary phase geometry. Solutions for any specific geometry must consequently be bracketed by the bounds. This may be regarded as a necessity (not sufficiency) test for any derived expression for effective elastic moduli."

1. **Brown W F, Jr.** Solid mixture permittivities. *J. Chem. Phys.* **23**:1514-7, 1955.
2. **Hashin Z & Shtrikman** S. A variational approach to the theory of the effective magnetic permeability of multiphase materials. *J. Appl. Phys.* **33**:3125-31, 1962.